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Conservation of e/m in General Relativity†

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Abstract

It is shown that when classical charged particles move under the influence of their mutual gravitational and electromagnetic forces, the charge-to-mass ratio of each particle remains constant in the particle's instantaneous rest frame, despite the fact that in general the particles are accelerating and therefore presumably radiation energy.

1. *Introduction*

It is a well-known consequence of Maxwell's theory that when charged particles are accelerated they radiate energy. In view of the relativistic requirement of the equivalence of energy and mass, one might wonder how elementary particles preserve their fixed e/m ratios when subject to external forces. This wonderment is further increased when one realizes that mass can be transported through the vacuum not only by electromagnetic radiation but also by gravitational radiation, whereas neither of these fields can transport charge through the vacuum.

At first sight one might think that in view of the fact that the sharply defined e/m ratios observed in nature are undoubtedly due to the (as yet unknown) quantum effects which determine the structure of the elementary particles, the conservation of e/m would require the introduction of quantum considerations for its elucidation. The purpose of this paper is to show that the above is not the case. We shall exhibit a simple and natural classical model of interacting massive, charged particles and shall demonstrate that it is a consequence of the Einstein–Maxwell field equations that the charge-to-mass ratio shall remain constant in the instantaneous rest-frame of each particle.

2. *The Classical Model*

The one simplification we shall make in setting up the model is to approximate the discrete particles by arbitrary continuous distributions of

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charge and mass. We do *not* assume that the charge and mass distributions are proportional for we want to approximate the interactions of particles with differing charge-to-mass ratios. One further simplifying assumption is that the motions of the particles is sufficiently non-turbulent that we can describe the world-lines of the particles by a continuous time-like unit vector field $U^\mu(X^\alpha)$.

In view of the fact that the only interactions which we wish to consider are gravitation and electromagnetism, we assume for the total stress tensor the form

$$T^{\mu\nu} = \rho_m(X^\alpha)\, U^\mu U^\nu + T_M^{\mu\nu} \tag{2.1}$$

where $\rho_m(X^\alpha)$ is proper mass density, $T_M^{\mu\nu}$ is the Maxwell stress tensor, defined in terms of the Maxwell field $F^{\mu\nu}$ by

$$T_M^{\mu\nu} = F_\alpha{}^\mu F^{\nu\alpha} - \tfrac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \tag{2.2}$$

and $U^\mu(X^\alpha)$ is a time-like unit vector field, i.e.

$$U^\mu U_\mu = 1 \tag{2.3}$$

The gravitational field $g_{\mu\nu}$ satisfies the Einstein field equations

$$G^{\mu\nu} = T^{\mu\nu} \tag{2.4}$$

and the electromagnetic field $F^{\mu\nu}$ satisfies the covariant Maxwell field equations

$$F_{\mu\nu;\sigma} + F_{\nu\sigma;\mu} + F_{\sigma\mu;\nu} = 0 \tag{2.5}$$

$$F^{\mu\nu}{}_{;\nu} = \rho_e(X^\alpha)\, U^\mu \tag{2.6}$$

where $\rho_e(X^\alpha)$ is the proper charge density. We wish to re-emphasize that we do not assume any relation between the distributions $\rho_m(X^\alpha)$ and $\rho_e(X^\alpha)$.

3. *Proof of Conservation*

With the model thus specified, we observe that it is generally covariant and gauge-invariant. Therefore, the corresponding Bianchi identities are valid. Namely, the covariant divergence of the left-hand sides of both equations (2.4) and (2.6) are identically zero. It follows that the right-hand sides of those two equations must also vanish. Thus, from equation (2.4) we obtain

$$(\rho_m U^\mu)_{;\mu} U^\nu + \rho_m U^\mu U^\nu{}_{;\mu} + T_{M\,;\mu}^{\mu\nu} = 0 \tag{3.1}$$

and from equation (2.6) we obtain

$$(\rho_e U^\mu)_{;\mu} = 0 \tag{3.2}$$

which is customarily called conservation of charge. Employing the Maxwell field equations (2.5) and (2.6) in the definition (2.2) we find the well-known result

$$T_{M\,;\mu}^{\mu\nu} = \rho_e U^\mu F_\mu{}^\nu \tag{3.3}$$

From the antisymmetry of $F_{\mu\nu}$ it follows immediately from equation (3.3)

$$U_\nu T^{\mu\nu}_{M\,;\mu} = 0 \tag{3.4}$$

Differentiating the unit-vector condition, equation (2.3) we also have

$$U_\nu U^\nu{}_{;\mu} = 0 \tag{3.5}$$

Thus contracting equation (3.1) with U_ν and employing equations (2.3), (3.4) and (3.5) we find

$$(\rho_m U^\mu)_{;\mu} = 0 \tag{3.6}$$

which we call conservation of matter. Mass, being equivalent to energy, cannot satisfy such a differential conservation law unless we also include the portion of the energy described by the Einstein pseudo-tensor contained in the gravitational field of the system.

If we now eliminate the term $U^\mu{}_{;\mu}$ between the two equations (3.2) and (3.6) we find

$$\frac{1}{\rho_m}\frac{d\rho_m}{ds} = \frac{1}{\rho_e}\frac{d\rho_e}{ds} \tag{3.7}$$

where we have introduced the notation $d\rho/ds \equiv \rho_{;\mu} U^\mu$ to indicate that the differentiation is being performed along the world-line of the particle, and in the instantaneous rest frame reduces to the ordinary time derivative.

Equation (3.7) can easily be rearranged to read

$$\frac{d}{ds}\left(\frac{\rho_e}{\rho_m}\right) = 0 \tag{3.8}$$

which is what we proposed to demonstrate. Note that the value of the ratio ρ_e/ρ_m as we move in directions normal to the world-lines of the particles is not determined, and can initially be assigned arbitrarily.

Incidentally, if we substitute equations (3.3) and (3.6) into equation (3.1) we obtain

$$\rho_m U^\nu{}_{;\mu} U^\mu + \rho_e U^\mu F_\mu{}^\nu = 0 \tag{3.9}$$

Thus, the trajectories of the particles are exactly what one would expect from generalizing the second law of Newton, with the Lorentz expression for the applied force, to a curved space. In the present model however, the law of motion, equation (3.9), was not assumed, but was rather deduced as a consequence of the theory.